N·H

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Discrete Mathematics 271 (2003) 295–302

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

The total chromatic number of graphs of even order and high degree

Dezheng Xie[a], Wanlian Yang[b]

[a] *Department of Mathematics, Chongqing Institute of Commerce, Chongqing 400067, China*

[b] *Department of Mathematics, Chongqing University, Chongqing 400041, China*

Received 2 February 2000; received in revised form 25 October 2002; accepted 15 November 2002

Abstract

For a given graph G, denote by G_Δ the subgraph of G induced by the vertices of maximum degree. The total chromatic number $\chi_T(G)$ of a graph G is the minimum number of colours needed to colour the edges and the vertices of G so that incident or adjacent elements have distinct colours. We show that if $G \neq K_2$ is of even order, G_Δ is a forest, and $\delta(G)+\Delta(G) \geqslant \frac{3}{2}(|V(G)|-1)$ then $\chi_T(G)=\Delta(G)+1$. We also show that for graphs G of even order and $\delta(G)+\Delta(G) \geqslant \frac{3}{2}|V(G)|-\frac{5}{2}$ we have that $\chi_T(G) \leqslant \Delta(G)+2$.

Keywords: Total chromatic number; Total colouring

1. Introduction

The graphs we shall consider are finite and simple. Let G be a graph. We denote its vertex set, edge set, complement, chromatic index, size (number of edges), minimum degree, maximum degree, and size of a maximum matching by $V(G)$, $E(G)$, $\bar{G}$, $\chi'(G)$, $e(G)$, $\delta(G)$, $\Delta(G)$, and $\alpha'(G)$, respectively. If $F \subseteq E(G)$, then $G-F$ is the graph obtained from G by deleting F from G. We denote by G_Δ the subgraph of G induced by the vertices of maximum degree. A vertex of maximum degree is called a major vertex, otherwise it is called minor. A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v.

E-mail address: xdzwp568@ctbu.edu.cn (D. Xie).

0012-365X/03/$ - see front matter
doi:10.1016/S0012-365X(03)00049-9

Given a graph G, a function $\pi : E(G) \cup V(G) \to N$ is called a (proper) *total colouring* if no two adjacent or incident elements are assigned the same colour from N. The *total chromatic number* of G, denoted $\chi_T(G)$, is the smallest positive integer k for which there exists a total colouring $\pi : E(G) \cup V(G) \to \{1, \ldots, k\}$.

From the definition of total chromatic number, it is clear that $\chi_T(G) \geqslant \Delta(G) + 1$. Behzad [1] and Vizing [13] independently made the following conjecture.

Total Colouring Conjecture (*TCC*). For any graph $G, \chi_T(G) \leqslant \Delta(G) + 2$.

This conjecture was proved for complete graphs, for graphs G having $\Delta(G) \leqslant 5$, for complete r-partite graphs, for graphs G having $\Delta(G) \geqslant \frac{3}{4}|V(G)|$, and for graphs G having $\Delta(G) \geqslant |V(G)| - 5$. For details, see [3,9–11,14,16].

If $\chi_T(G) = \Delta(G) + 1, G$ is said to be *type* 1 and if $\chi_T(G) = \Delta(G) + 2, G$ is said to be *type* 2. Hilton [8] gave a complete classification of graphs G of order $2n$ having $\Delta(G) = 2n - 1$ according to their total chromatic numbers. Yap [15] proved that if G is a graph of order $2n$ and G is type 1, then $e(\bar{G}) + \alpha'(\bar{G}) \geqslant n(2n - \Delta(G))$. Chen and Fu [2] gave a complete classification of graphs G of order $2n$ having $\Delta(G) = 2n - 2$ according to their total chromatic numbers, and Chew [4] proved that if G is a graph of odd order, minimum degree $\delta(G)$ and $r(G)$ vertices of maximum degree $\Delta(G)$ such that $\delta(G) + \Delta(G) \geqslant \frac{3}{2}|V(G)| + r(G) + \frac{5}{2}$, then $\chi_T(G) = \Delta(G) + 1$.

The main results of this paper are mentioned in the abstract.

2. Useful results

We begin by stating some useful results from the literature. The first is a result of Erdös and Pósa [5].

Lemma 1. *A graph G contains a matching of size at least* $\min\{\delta(G), [\frac{1}{2}\,|V(G)|]\}$.

The second is a result of Flandrin et al. [7], which generalises Dirac's theorem giving sufficient degree conditions for a graph to be Hamiltonian.

Lemma 2. *Let G be a 2-connected graph. If, for each set $\{v_1, v_2, v_3\}$ of three independent vertices,*

$$\sum_{i=1}^{3} d_G(v_i) \geqslant |V(G)| + \left| \bigcap_{i=1}^{3} N_G(v_i) \right|,$$

then G is Hamiltonian.

The third is a result of Hilton [8] and that of Yap [15].

Lemma 3. (1) *Let G be a graph of order $2n$ and $\Delta(G) = 2n - 1$. If $e(\bar{G}) + \alpha'(\bar{G}) \geqslant n$, then $\chi_T(G) = 2n$.*

(2) *For any graph G of order $2n$,*

$$e(\bar{G}) + \alpha'(\bar{G}) \geqslant n(2n + 1 - \chi_T(G)).$$

Corollary 1. *Let $G \neq K_2$ be a graph of even order. If G_Δ is a forest and $\Delta(G) = |V(G)| - 1$ then $\chi_T(G) = \Delta(G) + 1$.*

Proof. Let V_Δ be the set of major vertices of G. Since $G \neq K_2$, we may assume that $|V(G)| = 2n \geqslant 4$. Since G_Δ is a forest and $\Delta(G) = |V(G)| - 1$, it follows that $|V_\Delta| \leqslant 2$. Consequently, for each $v \in V(\bar{G}) - V_\Delta$, we have $d_{\bar{G}}(v) \geqslant 1$. Thus

$$e(\bar{G}) \geqslant \frac{2n-2}{2} = n - 1.$$

Now

$$e(\bar{G}) + \alpha'(\bar{G}) \geqslant n.$$

Hence, by Lemma 3, $\chi_T(G) = \Delta(G) + 1$. □

The following result is a corollary of Vizing's [12] theorem (see [6]).

Lemma 4. *Let G be a graph. If G_Δ is a forest, then $\chi'(G) = \Delta(G)$.*

Lemma 5. *Let G be a graph of order p, let $\Delta = \Delta(G)$ and let V_Δ be the set of major vertices of G. Suppose G_Δ is a forest and $p - 2 \geqslant \Delta \geqslant \frac{1}{2}p - 1$. Then $\bar{G}$ contains a matching $\{x_1y_1, \ldots, x_{p-\Delta-1}y_{p-\Delta-1}\}$ such that*

$$|V_\Delta - \{x_1, y_1, \ldots, x_{p-\Delta-1}, y_{p-\Delta-1}\}| \leqslant 1.$$

Proof. Since $\Delta \geqslant \frac{1}{2}p - 1$, we have $p - \Delta - 1 = \delta(\bar{G}) \leqslant \frac{1}{2}p$. Hence, by Lemma 1, $\bar{G}$ contains a matching of size $p - \Delta - 1$.

Let $M = \{x_1y_1, \ldots, x_{p-\Delta-1}y_{p-\Delta-1}\}$ be a matching of $\bar{G}$ such that $V_0 = \{x_1, y_1, \ldots, x_{p-\Delta-1}, y_{p-\Delta-1}\}$ has the maximum number of vertices in common with V_Δ, i.e. $|V_\Delta - V_0|$ is minimum.

Suppose $|V_\Delta - V_0| \geqslant 2$. Let $x, y \in V_\Delta - V_0$. We first prove that $|V_\Delta| \leqslant p - \Delta + 1$, which will be used later.

Suppose $|V_\Delta| > p - \Delta + 1$. Then for any $v \in V_\Delta$,

$$\begin{aligned} d_{G_\Delta}(v) &\geqslant \Delta - |V(G) - V_\Delta| = \Delta - (p - |V_\Delta|) = \Delta - p + |V_\Delta| \\ &\geqslant \Delta - p + (p - \Delta + 2) = 2, \end{aligned}$$

which contradicts the assumption that G_Δ is a forest. Next, from $p - 2 \geqslant \Delta$, it follows that

$$|V_0| = 2(p - \Delta - 1) \geqslant (p - \Delta + 1) - 1 \geqslant |V_\Delta| - 1.$$

Hence V_0 contains at least one minor vertex, say y_i.

Finally we consider two cases separately.

Case 1: $x_i \in V_\Delta$.

In this case, if $xy \notin E(G)$, then replacing x_iy_i by xy in M, we get a contradiction to the assumption that $|V_\Delta - V_0|$ is minimum. On the other hand, if $xy \in E(G)$, then x_i cannot be adjacent to both x and y, because G_Δ is a forest. Assume that $x_ix \notin E(G)$, then replacing x_iy_i by x_ix in M, we get another contradiction.

Case 2: $x_i \in V(G) - V_\Delta$.

It is clear that we can also assume that $xy \in E(G)$ and that each x_i and y_i is adjacent to both x and y. Let $z \in V(G)$ be such that $zx \notin E(G)$. It is clear that we can assume that $z \in V_0$. Let $z = z_j \in \{x_j, y_j\}$ and let z'_j be such that $\{z_j, z'_j\} = \{x_j, y_j\}$. Then $z'_j \in V_\Delta$ and $z'_j y \in E(G)$ otherwise we can easily obtain a contradiction. Moreover, since G_Δ is a forest, we have $z'_j x \notin E(G)$ and by case 1, we can also assume that $z_j \in V_\Delta$ and that $z_j y \in E(G)$. Thus V_0 contains at least $p - \Delta - 1$ major vertices $z_1, \ldots, z_{p-\Delta-1}$ which are not adjacent to x.

By symmetry, V_0 contains at least $p - \Delta - 1$ major vertices $w_1, \ldots, w_{p-\Delta-1}$ which are not adjacent to y and $\{z_1, \ldots, z_{p-\Delta-1}\} \cap \{w_1, \ldots, w_{p-\Delta-1}\} = \phi$. Consequently

$$|V_\Delta| \geqslant 2 + 2(p - \Delta - 1) > p - \Delta + 1$$

which is again a contradiction. $\square$

3. The main results

Theorem 1. *Let $G \neq K_2$ be a graph of even order and G_Δ be a forest. If $\delta(G) + \Delta(G) \geqslant \frac{3}{2}|V(G)| - \frac{3}{2}$ then $\chi_T(G) = \Delta(G) + 1$.*

Proof. Let $2n = |V(G)|, \Delta = \Delta(G), \delta = \delta(G)$ and let V_Δ be the set of major vertices of G. Since $G \neq K_2$ and G_Δ is a forest, from Corollary 1, we may assume that $2n \geqslant 4$ and $\Delta \leqslant 2n - 2$. Since G_Δ is a forest, $\Delta \geqslant \delta + 1$. Then

$$2\Delta \geqslant \delta + 1 + \Delta \geqslant \tfrac{3}{2}(2n) - \tfrac{1}{2} = 3n - \tfrac{1}{2}.$$

Hence

$$\Delta \geqslant \tfrac{3}{2}n - \tfrac{1}{4}. \tag{1}$$

Since $2n - 2 \geqslant \Delta \geqslant \frac{3}{2}n - \frac{1}{4}$, we have $n \geqslant 4$. Thus, we also assume that $2n \geqslant 8$.

By (1), we have $\Delta \geqslant n - 1 = \frac{1}{2}|V(G)| - 1$. By Lemma 5, there exists a subset $V_0 = \{x_1, y_1, \ldots, x_{2n-\Delta-1}, y_{2n-\Delta-1}\}$ of $V(G)$ such that $|V_\Delta - V_0| \leqslant 1$ and x_i and y_i are nonadjacent in G for $i = 1, \ldots, 2n - \Delta - 1$.

We now claim that there are $2n - \Delta - 1$ pairwise edge-disjoint matchings $F_1, \ldots, F_{2n-\Delta-1}$ of G such that each vertex of degree at least $n + 1$, except x_j and y_j, is F_j-saturated for $j = 1, \ldots, 2n - \Delta - 1$.

Suppose we have these matching $F_1, \ldots, F_{k-1}$, where $k \leqslant 2n - \Delta - 1$. Let $G_1 = G, \ldots, G_k = G - (F_1 \cup \cdots \cup F_{k-1})$ and let $H_k = G_k - \{x_k, y_k\}$. Form a new graph H'_k from H_k by pairwise joining all vertices with degree, in H_k, at most $n - 2$. It follows that

$$\begin{aligned} \delta(H_k) + \Delta(H_k) &\geqslant \delta(G_k) + \Delta(G_k) - 4 \\ &\geqslant \delta + \Delta - 2(2n - \Delta - 2) - 4 \\ &\geqslant 3n - \frac{3}{2} - 2(2n - \Delta - 2) - 4 \end{aligned}$$

$$= 2\Delta - n - \frac{3}{2}$$

$$\geqslant 2\left(\frac{3n}{2} - \frac{1}{4}\right) - n - \frac{3}{2} = 2n - 2. \qquad (2)$$

Hence

$$\delta(H'_k) + \Delta(H'_k) \geqslant \delta(H_k) + \Delta(H_k) \geqslant |V(H'_k)|.$$

Thus, H'_k is a connected graph.

Case 1: H'_k is 2-connected.

The graph H'_k has the property that all vertices of degree less than $n-1$ in H'_k are joined by an edge of H'_k. It follows that in any set of three independent vertices in H'_k, at most one vertex can have degree less than $n-1$. Thus, if $\{b_1, b_2, b_3\}$ is a set of independent vertices in H'_k, then

$$\sum_{i=1}^{3} d_{H'_k}(b_i) \geqslant 2(n-1) + \min_{i=1,2,3} \{d_{H'_k}(b_i)\}$$

$$\geqslant |V(H'_k)| + \left|\bigcap_{i=1}^{3} N_{H'_k}(b_i)\right|.$$

Since H'_k is also 2-connected, it follows from Lemma 2 that the graph H'_k is Hamiltonian. Let C_k be a Hamilton cycle in H'_k and let F_k be a maximum matching in H'_k such that $F_k \subseteq E(C_k) \cap E(G)$. By (2), there exists a vertex $v \in V(H_k)$ such that $d_{H_k}(v) \geqslant n-1 = \frac{1}{2}|V(H_k)|$, thus $F_k \neq \phi$. Since H'_k is obtained from H_k by the addition of some edges incident only with vertices of degree at most $n-2$ in H_k, all vertices of G of degree at least $n+1$, except x_k and y_k, are F_k-saturated.

Case 2: H'_k is 1-connected, but not 2-connected.

Let v_k be a cut vertex of H'_k and let $V'_k = \{v | d_{H'_k}(v) \geqslant n-1 = \frac{1}{2}|V(H'_k)| \text{ and } v \neq v_k\}$. By the property of the graph H'_k, it follows that $V'_k \neq \phi$. Let $H''_k = H'_k - \{v_k\}$. For each $w_1, w_2 \in V'_k$, $d_{H''_k}(w_1) \geqslant n-2$ and $d_{H''_k}(w_2) \geqslant n-2$, and so

$$d_{H''_k}(w_1) + d_{H''_k}(w_2) \geqslant 2n-4 = |V(H''_k)| - 1.$$

Thus, all vertices of V'_k occur in a single component of $H'_k - \{v_k\}$. Furthermore, all vertices $v \in V(H'_k) - \{v_k\}$ for which $d_{H'_k}(v) < n-1$ are joined by edges, and, thus, must also occur in a single component of $H'_k - \{v_k\}$. Since all vertices fall into one or the other of these classes, $H'_k - \{v_k\}$ contains exactly two components, the smaller of which is complete and the larger of which has the property that all vertices have degree (in $H'_k - \{v_k\}$) at least $n-2$ (and, thus, since this component can contain at most $2n-4$ vertices, is Hamiltonian).

Both components of $H'_k - \{v_k\}$ are Hamiltonian. Let H^*_k be the larger component of $H'_k - \{v_k\}$ and let C^*_k be a Hamilton cycle in H^*_k.

Suppose that C^*_k is an even cycle. Let F^*_k be a maximum matching in H^*_k such that $F^*_k \subseteq E(C^*_k)$. If $d_{H_k}(v_k) < n-1$, let $F_k = F^*_k \cap E(G)$. If $d_{H_k}(v_k) \geqslant n-1$, choose an edge $u_k v_k$ in H_k such that $u_k \in V(H_k) - V(H^*_k)$ and let $F_k = (F^*_k \cap E(G)) \cup \{u_k v_k\}$.

Suppose that C_k^* is an odd cycle. By (2), H_k is a connected graph. Thus, we can choose an edge $u_k'v_k$ in H_k such that $u_k' \in V(H_k^*)$ and let F_k' be the maximum matching in H_k^* such that $F_k' \subset E(C_k^*)$ and u_k' is not incident with an edge in F_k'. Let $F_k = (F_k' \cap E(G)) \cup \{u_k'v_k\}$.

It is the same as in case 1, all vertices of G of degree at least $n+1$, except x_k and y_k, are F_k-saturated.

The above cases show that the matchings $F_1, \ldots, F_{2n-\Delta-1}$ exist.

Let $G_{2n-\Delta} = G - (F_1 \cup \cdots \cup F_{2n-\Delta-1})$. Form a new graph G^* by adding a new vertex v^* to $G_{2n-\Delta}$ and joining v^* to each vertex of $V(G) - V_0$ by an edge. Now

$$\begin{aligned}|V(G) - V_0| &= 2n - 2(2n - \Delta - 1) = 2\Delta - 2n + 2\\ &\geqslant 2(\tfrac{3}{2}\,n - \tfrac{1}{4}) - 2n + 2\\ &= n + \tfrac{3}{2}.\end{aligned}$$

Thus $d_{G^*}(v^*) = 2\Delta - 2n + 2 \geqslant n + 2$. When we discuss maximum degree of G^*, we should consider the following two facts:

(i) Each vertex of G of degree at least $n+1$, except x_k and y_k, is F_k-saturated for $k = 1, \ldots, 2n - \Delta - 1$, and
(ii) $2\Delta - 2n + 2 \geqslant n + 2$.

Case 1: $v \in V_0$.

Since $\Delta - (2n - \Delta - 2) = 2\Delta - 2n + 2$, then each vertex v in V_0 has degree $d_{G^*}(v)$ at most $2\Delta - 2n + 2$ in G^*, and $d_G(v) = \Delta$ if and only if $d_{G^*}(v) = 2\Delta - 2n + 2$ for $v \in V_0$.

Case 2: $v \in V(G) - V_0$.

If $d_G(v) = \Delta$, then $d_{G^*}(v) = \Delta + 1 - (2n - \Delta - 1) = 2\Delta - 2n + 2$. If $d_G(v) < \Delta$, we have $d_{G^*}(v) \leqslant 2\Delta - 2n + 1$.

Thus $\Delta(G^*) = 2\Delta - 2n + 2$.

Since $|V_\Delta - V_0| \leqslant 1$, then there exists at most one major vertex of G in $V(G) - V_0$. Since G_Δ is a forest, it follows from the above discussion of the maximum degree of G^* that G_Δ^* is also a forest. By Lemma 4, there exists an edge colouring θ of G^* that uses $2\Delta - 2n + 2$ colours. We now form a total colouring π of G that uses $\Delta + 1$ colours as follows:

$$\begin{aligned}&\pi(e) = \theta(e) \quad \text{if } e \in E(G) \cap E(G^*),\\ &\pi(v) = \theta(vv^*) \quad \text{if } vv^* \in E(G^*),\\ &\pi(e) = k \quad \text{if } e \in F_k, \text{ for } k = 1, \ldots, 2n - \Delta - 1,\\ &\pi(x_k) = \pi(y_k) = k \quad \text{if } x_k, y_k \in V_0, \text{ for } k = 1, \ldots, 2n - \Delta - 1.\end{aligned}$$

It can be checked that π is indeed a total colouring of G. $\square$

Theorem 2. *Let G be a graph of even order and*

$$\delta(G) + \Delta(G) \geqslant \tfrac{3}{2}\,|V(G)| - \tfrac{5}{2}.$$

Then $\chi_{\mathrm{T}}(G) \leqslant \Delta(G) + 2$.

Proof. Let $|V(G)| = 2n$. If $\Delta(G) = 2n - 1$ or $2n = 2$, then the result is known in these cases (see [9]). Thus, we assume that $\Delta(G) \neq 2n - 1$ and $2n \neq 2$. Let $v \in V(G)$ be such that $d_G(v) = \Delta(G)$. Then there exists a vertex $u \in V(G)$ such that u and v is nonadjacent in G. Form a new graph G' by adding an edge uv to G. Then G'_Δ is a forest and

$$\delta(G') + \Delta(G') \geqslant \tfrac{3}{2}|V(G')| - \tfrac{3}{2}.$$

By Theorem 1, $\chi_{\mathrm{T}}(G') = \Delta(G') + 1$, it follows that

$$\chi_{\mathrm{T}}(G) \leqslant \chi_{\mathrm{T}}(G') = \Delta(G') + 1 = \Delta(G) + 2. \qquad \square$$

Corollary 2. *Let G be a graph of even order and such that* $\delta(G) \geqslant \frac{3}{4}\,|V(G)| - \frac{5}{4}$. *Then* $\chi_{\mathrm{T}}(G) \leqslant \Delta(G) + 2$. *If* G_Δ *is a forest then G is of type 1 unless* $G = K_2$.

Proof. From the hypothesis of the Corollary 2, we have

$$\delta(G) + \Delta(G) \geqslant 2\delta(G) \geqslant \tfrac{3}{2}\,|V(G)| - \tfrac{5}{2}.$$

By Theorem 2, $\chi_{\mathrm{T}}(G) \leqslant \Delta(G)+2$. Moreover, if G_Δ is also a forest, except for $G=K_2$, it follows that

$$\delta(G) + \Delta(G) \geqslant 2\delta(G) + 1 \geqslant \tfrac{3}{2}\,|V(G)| - \tfrac{3}{2}.$$

By Theorem 1, G is type 1. $\square$

Acknowledgements

The authors wish to thank the referees for valuable suggestions and criticisms.

References

[1] M. Behzad, Graphs and their chromatic numbers, Ph.D. Thesis, Michigan State University, 1965.

[2] B.L. Chen, H.L. Fu, Total colorings of graphs of order $2n$ having maximum degree $2n-2$, Graphs Combin. 8 (1992) 119–123.

[3] A.G. Chetwynd, A.J.W. Hilton, Z. Cheng, The total chromatic number of graphs of high minimum degree, J. London Math. Soc. 44 (2) (1991) 193–202.

[4] K.H. Chew, Total chromatic number of graphs of odd order and high degree, Discrete Math. 205 (1999) 39–46.

[5] P. Erdös, L. Pósa, On the maximal number of disjoint circuits in a graph, Publ. Math. Debrecen 9 (1962) 3–12.

[6] S. Fiorini, R.J. Wilson, Edge-colourings of Graphs, in: Research Notes in Mathematics, Vol. 16, Pitman, London, 1977.

[7] E. Flandrin, H.A. Jung, H. Li, Hamiltonism, degree sum and neighborhood intersections, Discrete Math. 90 (1991) 41–52.

[8] A.J.W. Hilton, A total chromatic number analogue of Plantholt's theorem, Discrete Math. 79 (1989/90) 169–175.
[9] A.J.W. Hilton, Recent results on the total chromatic number, Discrete Math. 111 (1993) 323–331.
[10] A.J.W. Hilton, H.R. Hind, The total chromatic number of graphs having large maximum degree, Discrete Math. 117 (1993) 127–140.
[11] A.V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Math. 162 (1996) 199–214.
[12] V.G. Vizing, On an estimate of the chromatic class of a p-graph, Diskret. Analiz 3 (1964) 25–30 (in Russian).
[13] V.G. Vizing, Some unsolved problems in graph theory, Uspehi Mat. Nauk 23 (1968) 117–134;
V.G. Vizing, Some unsolved problems in graph theory, Russian Math. Surveys 23 (1968) 125–142.
[14] H.P. Yap, Total Colouring of Graphs, in: Lecture Notes in Math., Vol. 1623, Springer, Berlin, 1986.
[15] H.P. Yap, Generalization of two results of Hilton on total-colourings of a graph, Discrete Math. 140 (1995) 245–252.
[16] H.P. Yap, Wang Jian-Fang, Zhang Zhongfu, Total chromatic number of graphs of high degree II, J. Austral. Math. Soc. Ser. A 53 (1992) 219–228.